Chapter 4 • Derivatives

Review of Prerequisite Skills

1. f. $\dfrac{4p^7 \times 6p^9}{12p^{15}} = \dfrac{24p^{16}}{12p^{15}}$

$= 2p$

i. $\dfrac{(3a^{-4})\left[2a^3(-b)^3\right]}{12a^5 - b^2} = -\dfrac{6a^{-1}b^3}{12a^5b}$

$= -\dfrac{b}{2a^6}$

6. d. $\dfrac{(x+y)(x-y)}{5(x-y)} \div \dfrac{(x+y)^3}{10}$

$= \dfrac{(x+y)}{5} \times \dfrac{10}{(x+y)^3}$

$= \dfrac{2}{(x+y)^2} \times x \neq y$

f. $\dfrac{x+1}{x-2} - \dfrac{x+2}{x+3} = \dfrac{(x+1)(x+3)-(x+2)(x-2)}{(x-2)(x+3)}$

$= \dfrac{x^2+4x+3-(x^2-4)}{(x-2)(x+3)}$

$= \dfrac{4x+7}{(x-2)(x+3)}$

9. c. $\dfrac{2+3\sqrt{2}}{3-4\sqrt{2}} = \dfrac{2+3\sqrt{2}}{3-4\sqrt{2}} \times \dfrac{3+4\sqrt{2}}{3+4\sqrt{2}}$

$= \dfrac{6+17\sqrt{2}+24}{9-32}$

$= \dfrac{30+17\sqrt{2}}{-23}$

$= -\dfrac{30+17\sqrt{2}}{23}$

d. $\dfrac{3\sqrt{2}-4\sqrt{3}}{3\sqrt{2}+4\sqrt{3}} = \dfrac{3\sqrt{2}-4\sqrt{3}}{3\sqrt{2}+4\sqrt{3}} \times \dfrac{3\sqrt{2}-4\sqrt{3}}{3\sqrt{2}-4\sqrt{3}}$

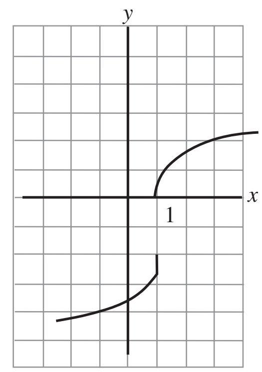
$= \dfrac{18-24\sqrt{6}+48}{18-48}$

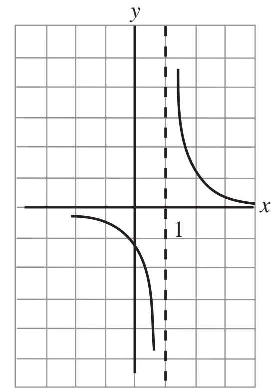
$= \dfrac{66-24\sqrt{6}}{-30}$

$= -\dfrac{11-4\sqrt{6}}{5}$

Exercise 4.1

3.





4. b. $f(x)x^2 + 3x + 1;\ a = 3$

$f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Since $a = 3$, $f(3+h) = (3+h)^2 + 3(3+h) + 1$

$= h^2 + 9h + 19$

$f(3) = 3^2 + 3(3) + 1$

$= 19$

Now $f(3+h) - f(3) = h^2 + 9h$

$= h(h+9)$

$f'(3) = \lim\limits_{h \to 0} \dfrac{h(h+9)}{h}$

$f'(3) = 9$

$f'(3) = \lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(3+h)^2 + 3(3+h) + 1 - (9+9+1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{6h + h^2 + 3h}{h} = \lim\limits_{h \to 0} \dfrac{9h + h^2}{h}$

$= 9$

c. $f(x)=\sqrt{x+1}$; $a=0$

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}\times\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$
$$=\lim_{h\to 0}\frac{h+1-1}{h\left(\sqrt{h+1}+1\right)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{h+1}+1}$$
$$f'(0)=\frac{1}{2}$$

5. a. $f(x)=x^2+3x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^2+3(x+h)-\left(x^2+3x\right)}{h}$$
$$=\lim_{h\to 0}\frac{2hx+h^2+3h}{h}$$
$$=\lim_{h\to 0}(2x+3+h)$$
$$f'(x)=2x+3$$

b. $f(x)=\dfrac{3}{x+2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{3}{x+h+2}-\frac{3}{x+2}}{h}$$
$$=\lim_{h\to 0}\frac{3x+6-3x-3h-6}{h(x+h+2)(x+2)}$$
$$=\lim_{h\to 0}\frac{-3}{(x+h+2)(x+2)}$$
$$f'(x)=\frac{-3}{(x+2)^2}$$

c.

$$f(x)=\sqrt{3x+2}$$
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+2}-\sqrt{3x+2}}{h}\times\frac{\sqrt{3x+3h+2}+\sqrt{3x+2}}{\sqrt{3x+3h+2}+\sqrt{3x+2}}$$
$$=\lim_{h\to 0}\frac{3x+3h+2-3x-2}{h\left(\sqrt{3x+3h+2}+\sqrt{3x+2}\right)}$$
$$f'(x)=\frac{3}{2\sqrt{3x+2}}$$

d. $f(x)=\dfrac{1}{x^2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$
$$=\lim_{h\to 0}\frac{x^2-x^2-2xh-h^2}{h(x+h)^2\left(x^2\right)}$$
$$=\lim_{h\to 0}\frac{-2x-h}{(x+h)^2\left(x^2\right)}$$
$$=\frac{-2x}{x^4}$$
$$f'(x)=-\frac{2}{x^3}$$

6. b. $y=\dfrac{x+1}{x-1}$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}}{h}$$
$$=\lim_{h\to 0}\frac{x^2+xh+x-x-h-1-x^2-xh+x-x-h+1}{h(x+h-1)(x-1)}$$
$$=\lim_{h\to 0}\frac{-2h}{h(x+h-1)(x-1)}$$
$$\frac{dy}{dx}=-\frac{2}{(x-1)^2}$$

7. $y = 2x^2 - 4x$

Since $y = f(x) = 2x^2 - 4x$

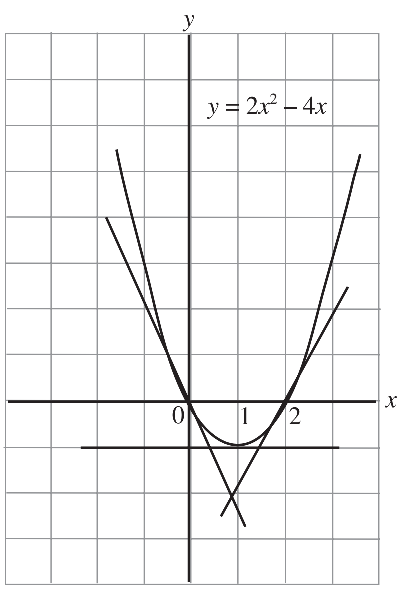
$$f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 4(x+h) \\ &= 2x^2 + 4hx + 2h^2 - 4x - 4h \\ f(x) &= 2x^2 - 4x \end{aligned}$$

$$f(x+h) - f(x) = 4hx + 2h^2 - 4h$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{h(4x+2h-4)}{h} \\ &= \lim_{h\to 0}(4x+2h-4) \\ f'(x) &= 4x - 4 \end{aligned}$$

Slopes of the tangents at $x = 0$, 1, and 2 are $f'(0) = -4$, $f'(1) = 0$, and $f'(2) = 4$.



8. $s(t) = -t^2 + 8t;\ t = 0,\ t = 4,\ t = 6$

$$v(t) = s'(t) = \lim_{h\to 0}\frac{s(t+h)-s(t)}{h}$$

$$\begin{aligned} s(t+h) &= -(t+h)^2 + 8(t+h) \\ &= -t^2 - 2ht - h^2 + 8t + 8h \\ s(t+h) - s(t) &= -2ht - h^2 + 8h \\ &= h(-2t - h + 8) \end{aligned}$$

$$\begin{aligned} v(t) &= \lim_{h\to 0}\frac{h(-2t-h+8)}{h} \\ &= \lim_{h\to 0}(-2t-h+8) \\ v(t) &= -2t + 8 \end{aligned}$$

Velocities at $t = 0$, 4, and 6 are $v(0) = 8$, $v(4) = 0$, and $v(6) = -4$.

9. $f(x) = \sqrt{x+1}$, parallel to $x - 6y + 4 = 0$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\times\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}} \\ &= \lim_{h\to 0}\frac{x+h+1-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} \\ f'(x) &= \frac{1}{2\sqrt{x+1}} \end{aligned}$$

The slope of the tangent to $f(x) = \sqrt{x+1}$ is parallel to $x - 6y + 4 = 0$.

$$\therefore f'(x) = \frac{1}{6}$$

$$\begin{aligned} \frac{1}{2\sqrt{x+1}} &= \frac{1}{6} \\ \sqrt{x+1} &= 3 \\ x &= 8 \end{aligned}$$

The point of tangency will be $(8, f(8)) = (8, 3)$.

The equation of the line will be $y - 3 = \frac{1}{6}(x - 8)$

or $x - 6y + 10 = 0$.

13. $\frac{1}{x}+\frac{1}{y}=1$ at (2, 2)

$$y=1-\frac{1}{x}=\frac{x-1}{x}$$

$$y=\frac{x}{x-1}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h}$$

$$=\lim_{h\to 0}\frac{x^2+xh-x-h-x^2-xh+x}{h(x-1)(x+h-1)}$$

$$=\lim_{h\to 0}\frac{-1}{(x-1)(x+h-1)}$$

$$=\frac{-1}{(x-1)^2}$$

At $x=2$, $f'(x)=-1$.

Slope of the tangent at (2, 2) is –1.

14. $f(x)=x|x|$

For $x<0$, $|x|=-x$ $\therefore f(x)=-x^2$

$x\ge 0$, $|x|=x$ $\therefore f(x)=x^2$

$\therefore f'(x)=-2x,\ x<0$

$f'(x)=2x,\ x\ge 0$

And $f'(x)$ exists for all $x\in R$ and $f'(0)=0$.

15. $f(a)=0,\ f'(a)=6$

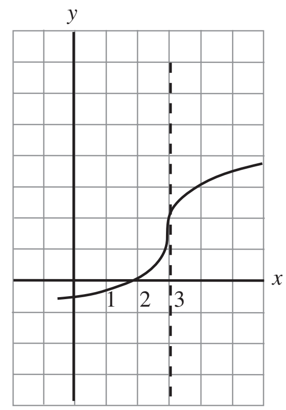
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=6$$

But $f(a)=0$

$$\therefore \lim_{h\to 0}\frac{f(a+h)}{h}=6$$

and $\lim_{h\to 0}\frac{f(a+h)}{2h}=3.$

16.



$f(x)$ is continuous.

$f(3)=2$

But $f'(3)=\infty$.

(Vertical tangent)

Exercise 4.2

2. **h.** $f(x)=\sqrt[3]{x}$

$$=x^{\frac{1}{3}}$$

$$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$$

k. $f(x)=\left(\frac{x}{2}\right)^4$

$$=\frac{x^4}{16}$$

$$f'(x)=\frac{4x^3}{16}$$

$$=\frac{x^3}{4}$$

3. **f.** $h(x)=(2x+3)(x+4)$

$$=2x^2+9x+12$$

$$h'(x)=4x+9$$

k. $s(t)=\dfrac{t^5-3t^2}{2t}$

$=\dfrac{t^4}{2}-\dfrac{3}{2}t$

$s'(t)=\dfrac{4t^3}{2}-\dfrac{3}{2}$

$=2t^3-\dfrac{3}{2}$

4. e. $y=3x^{\frac{2}{3}}-6x^{\frac{1}{3}}+x^{-\frac{1}{3}}$

$\dfrac{dy}{dx}=\dfrac{2}{3}\left(3x^{-\frac{1}{3}}\right)-\dfrac{1}{3}\left(6x^{-\frac{2}{3}}\right)-\dfrac{1x^{-\frac{4}{3}}}{3}$

$=2x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-\dfrac{1}{3}x^{-\frac{4}{3}}$

i. $y=20x^5+3\sqrt[3]{x}+17$

$=20x^5+3x^{\frac{1}{3}}+17$

$\dfrac{dy}{dx}=100x^4+3\times\dfrac{1}{3}x^{-\frac{2}{3}}$

$=100x^4+x^{-\frac{2}{3}}$

j. $y=\sqrt{x}+6\sqrt{x^3}+\sqrt{2}$

$=x^{\frac{1}{2}}+6x^{\frac{3}{2}}+\sqrt{2}$

$\dfrac{dy}{dx}=\dfrac{1}{2}x^{-\frac{1}{2}}+\dfrac{3}{2}\times 6x^{\frac{1}{2}}$

$=\dfrac{1}{2}x^{-\frac{1}{2}}+9x^{\frac{1}{2}}$

l. $y=\dfrac{1+\sqrt{x}}{x}$

$=\dfrac{1}{x}+\dfrac{\sqrt{x}}{x}$

$=x^{-1}+x^{-\frac{1}{2}}$

$\dfrac{dy}{dx}=-x^{-2}-\dfrac{1}{2}x^{-\frac{3}{2}}$

6. b. $f(x)=7-6x^{\frac{1}{2}}+5x^{\frac{2}{3}}$

$f'(x)=-3x^{-\frac{1}{2}}+\dfrac{10}{3}x^{-\frac{1}{3}}$

$f'(64)=-\dfrac{3}{\sqrt{64}}+\dfrac{10}{3\sqrt[3]{64}}$

$=-\dfrac{3}{8}+\dfrac{10}{12}$

$=\dfrac{11}{24}$

7. d. $y=\sqrt{16x^3}$

$=4x^{\frac{3}{2}}$

$\dfrac{dy}{dx}=6x^{\frac{1}{2}}$

At $(4, 32)$,

$\dfrac{dy}{dx}=6(2)$

$\dfrac{dy}{dx}=12.$

8. b. $y=2\sqrt{x}+5$

$y=2x^{\frac{1}{2}}+5$

$\dfrac{dy}{dx}=x^{-\frac{1}{2}}$

At $x=4$,

$\dfrac{dy}{dx}=\dfrac{1}{3}.$

d. $y=x^{-3}\left(x^{-1}+1\right)$

$=x^{-4}+x^{-3}$

$\dfrac{dy}{dx}=-4x^{-5}-3x^{-4}$

At $x=1$,

$\dfrac{dy}{dx}=-4-3$

$=-7.$

9. a. $y = 2x - \frac{1}{x}$ at $P(0.5, -1)$

$\frac{dy}{dx} = 2 + \frac{1}{x^2}$

Slope of the tangent at $x = \frac{1}{2}$ is $2 + 4 = 6$.

Equation $y + 1 = 6\left(x - \frac{1}{2}\right)$

$6x - y - 4 = 0$

b. $y = \frac{3}{x^2} - \frac{4}{x^3}$ at $P(-1, 7)$

$= 3x^{-2} - 4x^{-3}$

$\frac{dy}{dx} = -6x^{-3} + 12x^{-4}$

$\left.\frac{dy}{dx}\right]_{x=-1} = 6 + 12 = 18$

$y - 7 = 18(x + 1)$

$18x - y + 25 = 0$

c. $y = \sqrt{3x^3}$ at $P(3, 9)$

$\frac{dy}{dx} = \sqrt{3} \times \frac{3}{2}x^{\frac{1}{2}}$

$\left.\frac{dy}{dx}\right]_{x=3} = \sqrt{3} \times \frac{3}{2} \times \sqrt{3} = \frac{9}{2}$

$y - 9 = \frac{9}{2}(x - 3)$

$9x - 2y - 9 = 0$

d. $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$ at $P(1, 2)$

$= x + x^{-2}$

$\frac{dy}{dx} = 1 - 2x^{-3}$

Slope at $x = 1$ is –1.

$y - 2 = -1(x - 1)$

$x + y - 3 = 0$

e. $y = (\sqrt{x} - 2)(3\sqrt{x} + 8)$ at $(2, 2\sqrt{2} - 10)$

$= 3x + 2\sqrt{x} - 16$

$\frac{dy}{dx} = 3 + x^{-\frac{1}{2}}$

At $x = 4$, slope is $3 + \frac{1}{2} = \frac{7}{2}$.

Now, $y = \frac{7}{2}(x - 4)$

or $7x - 2y - 28 = 0$

f. $y = \frac{\sqrt{x} - 2}{\sqrt[3]{x}} = \frac{x^{\frac{1}{2}} - 2}{x^{\frac{1}{3}}} = x^{\frac{1}{6}} - 2x^{-\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{6}x^{-\frac{5}{6}} + \frac{2}{3}x^{-\frac{4}{3}}$

Slope at $x = 1$ is $\frac{1}{6} + \frac{2}{3} = \frac{5}{6}$.

Now, $y + 1 = \frac{5}{6}(x - 1)$; $5x - 6y - 11 = 0$.

10. A normal to the graph of a function at a point is a line that is perpendicular to the tangent at the given point.

$y = \frac{3}{x^2} - \frac{4}{x^3}$ at $P(-1, 7)$

Slope of the tangent is 18, therefore, the slope of the normal is $-\frac{1}{18}$.

Equation is $y - 7 = -\frac{1}{18}(x + 1)$.

$x + 18y - 125 = 0$

11. $y = \frac{3}{\sqrt[3]{x}} = 3x^{-\frac{1}{3}}$

Parallel to $x + 16y + 3 - 0$

Slope of the line is $-\frac{1}{16}$.

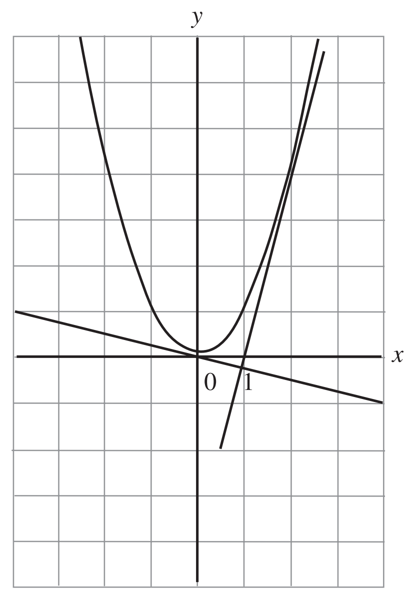
$\frac{dy}{dx} = -x^{-\frac{4}{3}}$

$\therefore x^{-\frac{4}{3}} = \frac{1}{16}$

$\frac{1}{x^{\frac{4}{3}}} = \frac{1}{16}$

$x^{\frac{4}{3}} = 16$

$x = (16)^{\frac{3}{4}} = 8$



12. $y = \frac{1}{x} = x^{-1}$: $y = x^3$

$\frac{dy}{dx} = -\frac{1}{x^2}$: $\frac{dy}{dx} = 3x^2$

Now, $-\frac{1}{x^2} = 3x^2$

$x^4 = -\frac{1}{3}$.

No real solution. They never have the same slope.

13. $y = x^2$, $\frac{dy}{dx} = 2x$

The slope of the tangent at $A(2, 4)$ is 4 and at

$B\left(-\frac{1}{8}, \frac{1}{64}\right)$ is $-\frac{1}{4}$.

Since the product of the slopes is –1, the tangents at

$A(2, 4)$ and $B\left(-\frac{1}{8}, \frac{1}{64}\right)$ will be perpendicular.

14. $y = -x^2 + 3x + 4$

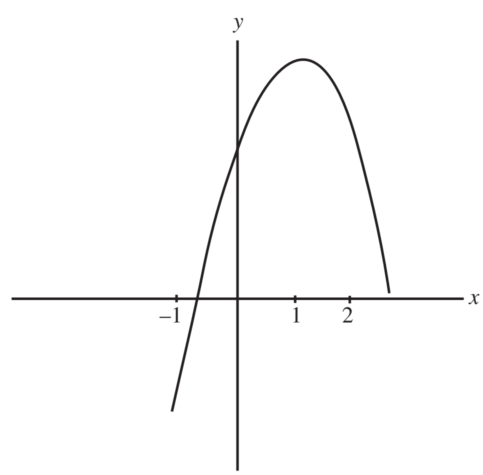
$\frac{dy}{dx} = -2x + 3$

For $\frac{dy}{dx} = 5$,

$5 = -2x + 3$

$x = -1$.

The point is $(-1, 0)$.



15. $$y = x^3 + 2$$

$$\frac{dy}{dx} = 3x^2, \text{ slope is } 12$$

$$\therefore x^2 = 4$$

$$x = 2 \quad \text{or} \quad x = -2$$

Points are $(2, 10)$ and $(-2, -6)$.

16. $y = \frac{1}{5}x^5 - 10x$, slope is 6

$$\frac{dy}{dx} = x^4 - 10 = 6$$

$$x^4 = 16$$

$$x^2 = 4 \quad \text{or} \quad x^2 = -4$$

$$x = \pm 2 \quad \text{non-real}$$

Tangents with slope 6 are at the points $\left(2, -\frac{68}{5}\right)$ and $\left(-2, \frac{68}{5}\right)$.

17. $y = 2x^2 + 3$

a. Equation of tangent from $A(2, 3)$:
If $x = a$, $y = 2x^2 + 3$.

Let the point of tangency be $P(a, 2a^2 + 3)$.

Now, $\frac{dy}{dx} = 4x$ and $\left.\frac{dy}{dx}\right]_{x=a} = 4a$.

The slope of the tangent is the slope of AP.

$$\therefore \frac{2a^2}{a-2} = 4a.$$

$$2a^2 = 4a^2 - 8a$$

$$2a^2 - 8a = 0$$

$$2a(a - 4) = 0$$

$$a = 0 \quad \text{or} \quad a = 4$$

Point (2, 3):

Slope is 0.	Slope is 16.
Equation of tangent is	Equation of tangent is
$y - 3 = 0$.	$y - 3 = 16(x - 2)$
	or $16x - y - 29 = 0$.

From the point $B(2, -7)$:

Slope of BP: $\frac{2a^2 + 10}{a - 2} = 4a$

$$2a^2 + 10 = 4a^2 - 8a$$

$$2a^2 - 8a - 10 = 0$$

$$a^2 - 4a - 5 = 0$$

$$(a - 5)(a + 1) = 0$$

$a = 5$	$a = -1$
Slope is $4a = 20$.	Slope is -4.
Equation is	Equation is
$y + 7 = 20(x - 2)$	$y + 7 = -4(x - 2)$
or $20x - y - 47 = 0$.	or $4x + y - 1 = 0$.

18. $ax - 4y + 21 = 0$ is tangent to $y = \frac{a}{x^2}$ at $x = -2$.

Therefore, the point of tangency is $\left(-2, \frac{a}{4}\right)$.

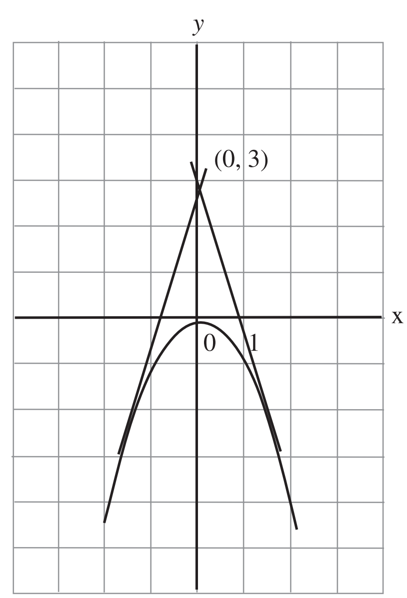
This point lies on the line, therefore,

$$a(-2) - 4\left(\frac{a}{4}\right) + 21 = 0$$

$$-3a + 21 = 0$$

$$a = 7.$$

22.



Let the coordinates of the points of tangency be $A(a, -3a^2)$.

$\frac{dy}{dx} = -6x$, slope of the tangent at A is $-6a$

Slope of PA: $\frac{-3a^2 - 3}{a} = -6a$

$$-3a^2 - 3 = -6a^2$$
$$3a^2 = 3$$

$a = 1$ or $a = -1$

Coordinates of the points at which the tangents touch the curve are $(1, -3)$ and $(-1, -3)$.

23. $y = x^3 - 6x^2 + 8x$, tangent at $A(3, -3)$

$$\frac{dy}{dx} = 3x^2 - 12x + 8$$

$$\left.\frac{dy}{dx}\right]_{x=3} = 27 - 36 + 8 = -1$$

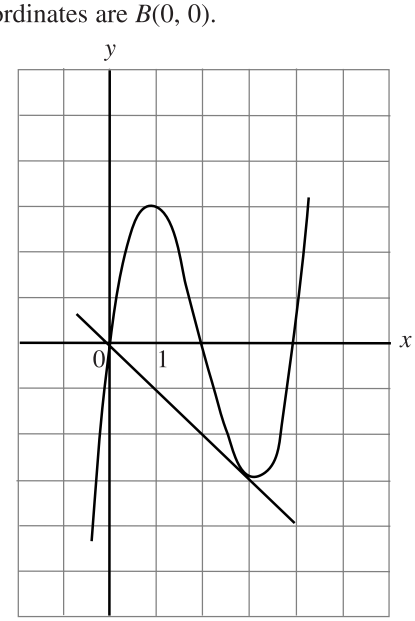
The slope of the tangent at $A(3, -3)$ is -1.

Equation will be

$$y + 3 = -1(x - 3)$$
$$y = -x.$$
$$-x = x^3 - 6x^2 + 8x$$
$$x^3 - 6x^2 + 9x = 0$$
$$x(x^2 - 6x + 9) = 0$$
$$x(x-3)^2 = 0$$

$x = 0$ or $x = 3$

Coordinates are $B(0, 0)$.



24. $\sqrt{x} + \sqrt{y} = 1$

$P(a, b)$ is on the curve, therefore $a \geq 0,\ b \geq 0$.

$$\sqrt{y} = 1 - \sqrt{x}$$
$$y = 1 - 2\sqrt{x} + x$$
$$\frac{dy}{dx} = -\frac{1}{2} \bullet 2x^{-\frac{1}{2}} + 1$$

At $x = a$, slope is $-\frac{1}{\sqrt{a}} + 1 = -\frac{1 + \sqrt{a}}{\sqrt{a}}$.

But $\sqrt{a} + \sqrt{b} = 1$

$$-\sqrt{b} = \sqrt{a} - 1.$$

Therefore, slope is $-\frac{\sqrt{b}}{\sqrt{a}} = -\sqrt{\frac{b}{a}}$.

25. $f(x) = x^n,\ f'(x) = nx^{n-1}$

Slope of the tangent at $x = 1$ is $f'(1) = n$,
The equation of the tangent at
(1, 1) is:

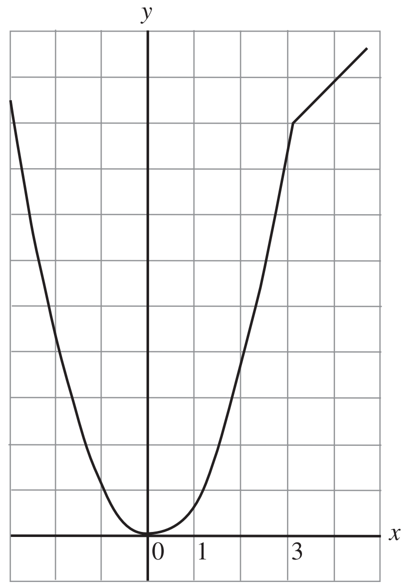
$$y - 1 = n(x - 1)$$
$$nx - y - n + 1 = 0$$

Let $y = 0$, $nz = n - 1$

$$x = \frac{n-1}{n} = 1 - \frac{1}{n}.$$

The x-intercept is $1 - \frac{1}{n}$ as $n \to \infty$, and $\frac{1}{n} \to 0$, and the x-intercept approaches 1 as $n \to \infty$, the slope of the tangent at (1, 1) increases without bound, and the tangent approaches a vertical line having equation $x - 1 = 0$.

26. a.



$$f(x)=\begin{cases} x^2, & x<3 \\ x+6, & x\ge 3\end{cases} \qquad f'(x)=\begin{cases} 2x, & x<3 \\ 1, & x\ge 3\end{cases}$$

$f'(3)$ does not exist.

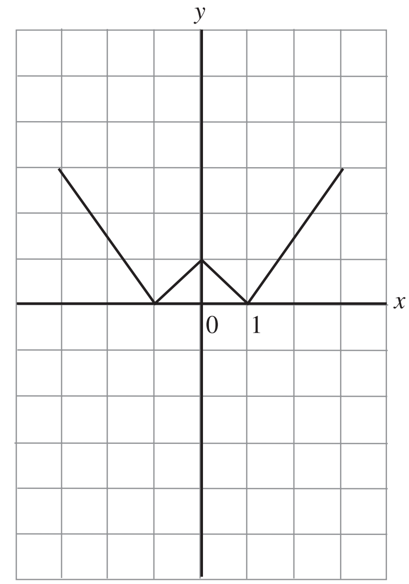
b.

$$f(x)=\begin{cases} 3x^2-6, & x<-\sqrt{2} \text{ or } x>\sqrt{2} \\ 6-3x^2, & -\sqrt{2}<x<\sqrt{2}\end{cases}$$

$$f'(x)=\begin{cases} 6x, & x<-\sqrt{2} \text{ or } x>\sqrt{2} \\ -6x, & -\sqrt{2}\le x\le \sqrt{2}\end{cases}$$

$f'(\sqrt{2})$ and $f'(-\sqrt{2})$ do not exist.

c.



$$f(x)=\begin{cases} x-1, x\ge 1 & \text{since} & |x-1|=x-1 \\ 1-x, 0\le x<1 & \text{since} & |x-1|=1-x \\ x+1, -1<x<0 & \text{since} & |-x-1|=x+1 \\ -x-1, x\le -1 & \text{since} & |-x-1|=-x-1\end{cases}$$

$$f'(x)=\begin{cases} 1, & x>1 \\ -1, & 0<x<1 \\ 1, & -1<x<0 \\ -1, & x<-1\end{cases}$$

$f'(0)$, $f'(-1)$, and $f'(1)$ do not exist.

Exercise 4.3

2. c. $y=(1-x^2)^4(2x+6)^3$

$$\frac{dy}{dx}=4(1-x^2)^3(-2x)(2x+6)^3+(1-x^2)^4 3(2x+6)^2(2)$$

$$=-8x(1-x^2)^3(2x+6)^3+6(1-x^2)^4(2x+6)$$

4. e. $y=x^3(3x+7)^2$

$$\frac{dy}{dx}=3x^2(3x+7)^2+x^3 6(3x+7)$$

At $x=-2$,

$$\frac{dy}{dx}=12(1)^2+(-8)(6)(1)$$

$$=12-48$$

$$=-36.$$

f. $y=(2x+1)^5(3x+2)^4, \quad x=-1$

$$\frac{dy}{dx}=5(2x+1)^4(2)(3x+2)^4+(2x+1)^5 4(3x+2)^3(3)$$

$$\left.\frac{dy}{dx}\right]_{x=-1}=5(-1)^4(2)(-1)^4+(-1)^5(4)(-1)^3(3)$$

$$=10+12$$

$$=22$$

h. $y=3x(x-4)(x+3),\ x=2$

$$\frac{dy}{dx}=3(x-4)(x+3)+3x(x+3)+3x(x-4)$$

At $x=2$,

$$\frac{dy}{dx}=3(-2)(5)+6(5)+6(-2)$$

$$=-30+30-12$$

$$=-12.$$

5. Tangent to $y=(x^3-5x+2)(3x^2-2x)$ at (1, –2).

$$\frac{dy}{dx}=(3x^2-5)(3x^2-2x)+(x^3-5x+2)(6x-2)$$

$$\left.\frac{dy}{dx}\right]_{x=1}=(-2)(1)+(-2)(4)$$

$$=-2\bullet-8$$

$$=-10$$

Slope of the tangent at (1, –2) is –10.
The equation is $y+2=-10(x-1)$; $10x+y-8=0$.

6. b. $y=(x^2+2x+1)(x^2+2x+1)$

$$\frac{dy}{dx}=2(x^2+2x+1)(2x+2)$$

$$(x^2+2x+1)(2x+2)=0$$

$$2(x+1)(x+1)(x+1)=0$$

$$x=-1$$

Point of horizontal tangency is (–1, 0).

7. b. $y=x^2(3x^2+4)^2(3-x^3)^4$

$$\frac{dy}{dx}=2x(3x^2+4)^2(3-x^3)^4$$
$$+x^2[2(3x^2+4)(6x)](3-x^3)^4$$
$$+x^2(3x^2+4)^2\left[4(3-x^3)^3(-3x^2)\right]$$

8. Determine the point of tangency, and then find the negative reciprocal of the slope of the tangency. Use this information to find the equation of the normal.

$$h(x)=2x(x+1)^3(x^2+2x+1)^2$$

$$h'(x)=2(x+1)^3(x^2+2x+1)^2+2x3(x+1)^2(x^2+2x+1)^2$$
$$+2x(x+1)^3 2(x^2+2x+1)(2x+1)$$

$$h'(-2)=2(-1)^3(1)^2+2(-2)(3)(-1)^2(1)^2+2(-2)(-1)^3(2)(1)(-2)$$

$$=-2-12-16$$

$$=-30$$

9. a. $f(x)=g_1(x)g_2(x)g_3(x)\ \cdots\ g_{n-1}(x)g_n(x)$

$$f'(x)=g_1{}^1(x)g_2(x)g_3(x)\ \cdots\ g_{n-1}(x)g_n(x)$$
$$+g_1(x)g_2{}^1(x)g_3(x)\ \cdots\ g_{n-1}(x)g_n(x)$$
$$+g_1(x)g_2(x)g_3{}^1(x)\ \cdots\ g_{n-1}(x)g_n(x)$$
$$+\ \ldots$$
$$+g_1(x)g_2(x)g_3(x)\ \cdots\ g_{n-1}(x)g_n(x)$$

b. $f(x)=(1+x)(1+2x)(1+3x)\ldots(1+nx)$

$$f'(x)=1(1+2x)(1+3x)\ \ldots(1+nx)$$
$$+(1+x)(2)(1+3x)\ \cdots\ (1+nx)$$
$$+(1+x)(1+2x)(3)\ \cdots\ (1+nx)$$
$$+\ \ldots$$
$$+(1+x)(1+2x)(1+3x)\ldots(n)$$

$$\therefore f'(0)=1(1)(1)(1)\ \cdots\ (1)$$
$$+1(2)(1)(1)\ldots(1)$$
$$+1(1)(3)(1)\ldots(1)$$
$$+\ \ldots$$
$$+(1)(1)(1)\ \ldots\ (n)$$
$$=1+2+3+\ \ldots+n$$

$$f'(0)=\frac{n(n+1)}{2}$$

10. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b \qquad (1)$

Horizontal tangent at $(-1, -8)$

$f'(x) = 0$ at $x = -1$

$2a + b = 0$

Since $(2, 19)$ lies on the curve,

$4a + 2b + c = 19. \qquad (2)$

Since $(-1, -8)$ lies on the curve,

$a - b + c = -8. \qquad (3)$

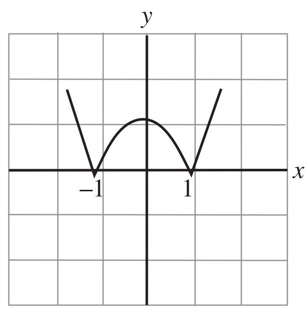
$$\begin{aligned} 4a + 2b + c &= 19 \\ -3a - 3b &= -27 \\ a + b &= 9 \\ 2a + b &= 0 \\ -a &= 9 \end{aligned}$$

$a = -9, \quad b = 18$

$$\begin{aligned} -9 - 18 + c &= -8 \\ c &= 19 \end{aligned}$$

The equation is $y = -9x^2 + 18x + 19$.

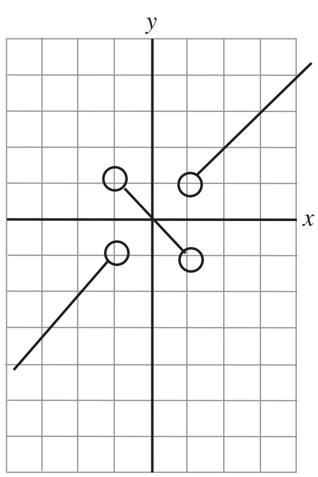
11.



a. $x = 1$ or $x = -1$

b. $f'(x) = 2x, x < -1$ or $x > 1$

$f'(x) = -2x, -1 < x < 1.$



c. $f'(-2) = 2(-2) = -4$

$f'(0) = -2(0) = 0$

$f'(3) = 2(3) = 6$

12. $y = \dfrac{16}{x^2} - 1$

$\dfrac{dy}{dx} = -\dfrac{32}{x^3}$

Slope of the line is 4.

$$\begin{aligned} -\frac{32}{x^3} &= 4 \\ 4x^3 &= -32 \\ x^3 &= -8 \\ x &= -2 \\ y &= \frac{16}{4} - 1 \\ &= 3 \end{aligned}$$

Point is at $(-2, 3)$.

Find intersection of line and curve:

$$\begin{aligned} 4x - y + 11 &= 0 \\ y &= 4x + 11. \end{aligned}$$

Substitute,

$$\begin{aligned} 4x + 11 &= \frac{16}{x^2} - 1 \\ 4x^3 + 11x^2 &= 16 - x^2 \end{aligned}$$

or $4x^3 + 12x^2 - 16 = 0.$

Let $x = -2$.

$$\begin{aligned} \text{R.S.} &= 4(-2)^3 + 12(-2)^2 - 16 \\ &= 0 \end{aligned}$$

Since $x = -2$ satisfies the equation, therefore it is a solution.

When $x = -2$, $y = 4(-2) + 11 = 3.$

Intersection point is $(-2, 3)$. Therefore, the line is tangent to the curve.

Exercise 4.4

4. g. $y = \dfrac{x(3x+5)}{(1-x^2)} = \dfrac{3x^2+5x}{1-x^2}$

$$\frac{dy}{dx} = \frac{(6x+5)(-x^2)-(3x^2+5x)(-2x)}{(1-x^2)^2}$$

$$= \frac{-6x^3-5x^2+6x+5+6x^3+10x^2}{(1-x^2)^2}$$

$$= \frac{5x^2+6x+5}{(1-x^2)^2}$$

5. c. $y = \dfrac{x^2-25}{x^2+25}, \quad x = 2$

$$\frac{dy}{dx} = \frac{2x(x^2+25)-(x^2-25)(2x)}{(x^2+25)^2}$$

$$\left.\frac{dy}{dx}\right]_{x=2} = \frac{4(29)-(-21)(4)}{(29)^2}$$

$$= \frac{116+84}{29^2}$$

$$= \frac{200}{841}$$

d. $y = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}, x = 4$

$$= \frac{x^2+3x+2}{x^2-3x+2}$$

$$\frac{dy}{dx} = \frac{(2x+3)(x^2-3x+2)-(x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2}$$

At $x = 4$:

$$\frac{dy}{dx} = \frac{(11)(6)-(30)(5)}{(9)(4)}$$

$$= -\frac{84}{36}$$

$$= -\frac{7}{3}$$

6. $y = \dfrac{x^3}{x^2-6}$

$$\frac{dy}{dx} = \frac{3x^2(x^2-6)-x^3(2x)}{(x^2-6)^2}$$

At $(3, 9)$:

$$\frac{dy}{dx} = \frac{3(9)(3)-(27)(6)}{(3)^2}$$

$$= 9-18$$

$$= -9$$

The slope of the tangent to the curve at $(3, 9)$ is -9.

7. $y = \dfrac{3x}{x-4}$

$$\frac{dy}{dx} = \frac{3(x-4)-3x}{(x-4)^2} = -\frac{12}{(x-4)^2}$$

Slope of the tangent is $-\dfrac{12}{25}$.

Therefore, $\dfrac{12}{(x-4)^2} = \dfrac{12}{25}$

$\therefore x-4 = 5$ or $x-4 = -5$

$x = 9$ or $x = -1$.

Points are $\left(9, \dfrac{27}{5}\right)$ and $\left(-1, \dfrac{3}{5}\right)$.

8. $f(x) = \dfrac{5x+2}{x+2}$

$$f'(x) = \frac{(x+2)(5)-(5x+2)(1)}{(x+2)^2}$$

$$f'(x) = \frac{8}{(x+2)^2}$$

Since $(x+2)^2$ is positive or zero for all $x \in R$,

$\dfrac{8}{(x+2)^2} > 0$ for $x \neq -2$. Therefore, tangents to the

graph of $f(x) = \dfrac{5x+2}{x+2}$ do not have a negative slope.

9. b. $y=\dfrac{x^2-1}{x^2+x-2}$

$=\dfrac{(x-1)(x+1)}{(x+2)(x-1)}$

$=\dfrac{x+1}{x+2},\ x\neq 1$

$\dfrac{dy}{dx}=\dfrac{(x+2)-(x+1)}{(x+2)^2}$

$=\dfrac{1}{(x+2)^2}$

Curve has horizontal tangents when $\dfrac{dy}{dx}=0$.

No value of x will give a horizontal slope, therefore, there are no such tangents.

10. $p(t)=1000\left(1+\dfrac{4t}{t^2+50}\right)$

$p'(t)=1000\left(\dfrac{4(t^2+50)-4t(2t)}{(t^2+50)^2}\right)$

$=\dfrac{1000(200-4t^2)}{(t^2+50)^2}$

$p'(1)=\dfrac{1000(196)}{(51)^2}\doteq 75.36$

$p'(2)=\dfrac{1000(184)}{54^2}\doteq 63.10$

Population is growing at a rate of 75.4 bacteria per hour at t = 1 and at 63. One bacteria per hour at t = 2.

12. a. $s(t)=\dfrac{10(6-t)}{t+3},\ 0\le t\le 6\ t=0,\ s(0)=20$

The boat is initially 20 m from the dock.

b. $v(t)=s'(t)=10\left[\dfrac{(t+3)(-1)-(6-t)(1)}{(t+3)^2}\right]$

$v(t)=\dfrac{-90}{(t+3)^2}$

At $t=0,\ v(0)=-10$, the boat is moving towards the dock at a speed of 10 m/s. When $s(t)=0$, the boat will be at the dock.

$\dfrac{10(6-t)}{t+3}=0,\ t=6$.

$v(6)=\dfrac{-90}{9^2}=-\dfrac{10}{9}$

The speed of the boat when it bumps into the dock is $\dfrac{10}{9}$ m/s.

13. $f(x)=\dfrac{ax+b}{cx+d},\ x\neq -\dfrac{d}{c}$

$f'(x)=\dfrac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$

$f'(x)=\dfrac{ad-bc}{(cx+d)^2}$

For the tangents to the graph of $y=f(x)$ to have positive slopes, $f'(x)>0$. $(cx+d)^2$ is positive for all $x\in R$. $ad-bc>0$ will ensure each tangent has a positive slope.

Exercise 4.5

4. b. If $g(x)=5x-1$ and $f(x)=\sqrt{x}$,

then $h(x)=f(g(x))$

$=f(5x-1)$

$f(x)=\sqrt{5x-1}.$

e. $h(x)=x^4+5x^2+6$

$=(x^2+2)(x^2+3)$

$=(x^2+2)(x^2+2+1)$

If $g(x)=x^2+2$ and $f(x)=x(x+1)$,

then $h(x)=f(g(x))$

$=g(x)(g(x)+1)$

$=(x^2+2)(x^2+2+1)$

$h(x)=x^4+5x^2+6.$

5. $f(x)=\sqrt{2-x}$ and $f(g(x))=\sqrt{2-x^3}$

$$f(g(x))=\sqrt{2-g(x)}=\sqrt{2-x^3}$$
$$g(x)=x^3$$

6. $g(x)=\sqrt{x},\ f(g(x))=\left(\sqrt{x}+7\right)^2$

$$f(x)=(x+7)^2$$

7. $g(x)=x-3,\ f(g(x))=x^2$

$$\therefore f(x-3)=x^2$$
$$f(x-3)=[(x-3)+3]^2$$
$$\therefore f(x)=(x+3)^2$$

Or, since $g(x)$ is linear and $f(g(x))$ is quadratic, $f(x)$ is a quadratic function.

Let $f(x)=ax^2+bx+c$.

$$\therefore f(g(x))=a(x-3)^2+b(x-3)+c=x^2$$
$$ax^2-bax+ga+bx-3b+c=x^2$$
$$ax^2+x(b-6a)+9a-3b+c=x^2$$

Equating coefficients:

$$a=1$$
$$b-6a=0 \qquad b=6$$
$$9a-3b+c=0 \qquad c=9.$$
$$\therefore f(x)=x^2+6x+9$$
$$f(x)=(x+3)^2$$

8. $f(x)=x^2,\ f(g(x))=x^2+8x+16$

But $f(g(x))=[g(x)]^2$.

$$\therefore [g(x)]^2=x^2+8x+16=(x+4)^2$$
$$g(x)=x+4 \text{ or } g(x)=-x-4$$

9. $f(x)=x+4,\ g(x)=(x-2)^2$

and $f(g(u(x)))=4x^2-8x+8$

$$g(u(x))=(u(x)-2)^2$$

and $f(g(u(x)))=f\left((u(x)-2)^2\right)$

$$=(u(x)-2)^2=4$$
$$=(u(x))^2-4u(x)+8$$

Since $f(g(u(x)))$ is quadratic, $u(x)$ must be linear.

Let $u(x)=ax+b$.

Now,

$$(ax+b)^2-4(ax+b)+8=4x^2-8x+8$$
$$a^2=4,\quad a=2,\ \text{or}\qquad a=-2$$
$$2ab-4a=-8,\ b=0,\ \text{or}\ -4b+8=-8$$
$$b=4.$$
$$\therefore u(x)=2x \text{ or } u(x)=-2x+4$$

10. $f(x)=\dfrac{1}{1-x},\ g(x)=1-x$

a. $g(f(x))=g\left(\dfrac{1}{1-x}\right)$

$$=1-\frac{1}{1-x}$$
$$=\frac{1-x-1}{1-x}$$
$$=-\frac{x}{1-x}=\frac{x}{x-1}$$

b. $f(g(x))=f(1-x)$

$$=\frac{1}{1-(1-x)}$$
$$=\frac{1}{x}$$

11. $f(g(x)) = g(f(x))$

$$3(x^2+2x-3)+5 = (3x+5)^2+2(3x+5)-3$$
$$3x^2+6x-9+5 = 9x^2+30x+25+6x+10-3$$
$$3x^2+6x-4 = 9x^2+36x+32$$
$$6x^2+30x+36 = 0$$
$$x^2+5x+6 = 0$$
$$(x+3)(x+2) = 0$$
$$x=-3 \text{ or } x=-2$$

12. a. $f(x) = 2x-7$

$$f^{-1}(x) = \frac{x+7}{2}$$
$$f \circ f^{-1} = f\left(\frac{x+7}{2}\right)$$
$$= 2\left(\frac{x+7}{2}\right)-7$$
$$= x$$

$$f^{-1} \circ f = f^{-1}(2x-7)$$
$$= \frac{2x-7+7}{2}$$
$$= x$$

b. $f \circ g = f(5-2x)$
$$= 2(5-2x)-7$$
$$= 10-4x-7$$
$$= 3-4x$$

$$(f \circ g)^{-1} = \frac{3-x}{4}$$

Note: $g^{-1}(x) = \dfrac{5-x}{2}$.

$$g^{-1} \times f^{-1} = g^{-1}\left(\frac{x+7}{2}\right)$$
$$= \frac{5-\frac{x+7}{2}}{2}$$
$$= \frac{10-x-7}{4}$$
$$= \frac{3-x}{4}$$

Exercise 4.6

3. f. $y = \dfrac{3}{9-x^2} = 3(9-x^2)^{-1}$

$$\frac{dy}{dx} = \frac{6x}{(9-x^2)^2}$$

i. $y = \left(\dfrac{1+\sqrt{x}}{\sqrt[3]{x^2}}\right)^3 = (1+\sqrt{x})^3 (x^2)^{-1}$

$$\frac{dy}{dx} = 3\left(\frac{1+\sqrt{x}}{3\sqrt{x^2}}\right)^2 \times \left[\frac{x^{\frac{2}{3}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \left(1+x^{\frac{1}{2}}\right)\frac{2}{3}x^{-\frac{1}{3}}}{\left(x^{\frac{2}{3}}\right)^2}\right]$$

$$\frac{dy}{dx} = 3\left(\frac{1+\sqrt{x}}{3\sqrt{x^2}}\right)^2 \left[\frac{\dfrac{x^{\frac{2}{3}}}{2x^{\frac{1}{2}}} - \dfrac{2\left(1+x^{\frac{1}{2}}\right)}{3x^{\frac{1}{3}}}}{x^{\frac{4}{3}}}\right]$$

$$= 3\left(\frac{1+\sqrt{x}}{3\sqrt{x^2}}\right)^2 \left[\frac{3x-4x^{\frac{1}{2}}-4x}{6x^{\frac{5}{6}}x^{\frac{4}{3}}}\right]$$

$$= 3\left(\frac{1+\sqrt{x}}{3\sqrt{x^2}}\right)^2 \left[\frac{-x-4\sqrt{x}}{6x^{\frac{13}{6}}}\right]$$

$$\frac{-3(1+\sqrt{x})^2}{x^{\frac{4}{3}}} \times \frac{x^{\frac{1}{2}}(4+\sqrt{x})}{6x^{\frac{13}{6}}} = \frac{-(1+\sqrt{x})^2(4+\sqrt{x})x^{\frac{3}{6}}}{2x^{\frac{21}{6}}}$$

$$= -\left(\frac{1-\sqrt{x}}{3\sqrt{x^2}}\right)^2 \left[\frac{x+4\sqrt{x}}{2x^2 6\sqrt{x}}\right] = -\frac{(1+\sqrt{x})^2(4+\sqrt{x})}{2x^3}$$

4. f. $y=\dfrac{(2x-1)^2}{(x-2)^3}$

$$\frac{dy}{dx}=\frac{2(x-2)^3(2x-1)(2)-3(2x-1)^2(x-2)^2}{(x-2)^6}$$
$$=\frac{(x-2)^2(2x-1)[4(x-2)-3(2x-1)]}{(x-2)^6}$$
$$=-\frac{(x-2)^2(2x-1)(2x+5)}{(x-2)^6}$$

k.
$$s=(4-3t^3)^4(1-2t)^6$$
$$\frac{ds}{dt}=4(4-3t^3)^3(-9t^2)(1-2t)^6+6(4-3t^3)^4(1-2t)^5(-2)$$
$$=12(4-3t^3)^3(1-2t)^5[-3t^2(1-2t)-(4-3t^3)]$$
$$=12(4-3t^3)^3(1-2t)^5(9t^3-3t^2+4)$$
$$=12(4-3t^3)^3(1-2t)^5(9t^3-3t^2-4)$$

l. $h(x)=\dfrac{\sqrt{1-x^2}}{1-x}$
$$h'(x)=\frac{\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)(1-x)-\sqrt{1-x^2}(-1)}{(1-x)^2}$$
$$=\left(\frac{-x(1-x)}{\sqrt{1-x^2}}+\frac{1-x^2}{\sqrt{1-x^2}}\right)\frac{1}{(1-x)^2}$$
$$=\frac{1-x}{\sqrt{1-x^2}(1-x)^2}=\frac{1}{(1-x)\sqrt{1-x^2}}$$

6. $y=(1+x^3)^2$ $\qquad y=2x^6$

$\dfrac{dy}{dx}=2(1+x^3)(3x^2)$ $\qquad \dfrac{dy}{dx}=12x^5$

For the same slope,
$$6x^2(1+x^3)=12x^5$$
$$6x^2+6x^5=12x^5$$
$$6x^5-6x^2=0$$
$$6x^2(x^3-1)=0$$
$$x=0 \quad \text{or} \quad x=1.$$
Curves have the same slope at $x=0$ and $x=1$.

8. $y=(x^3-7)^5$ at $z=2$
$$\frac{dy}{dx}=5(x^3-7)^4(3x^2)$$
$$\left.\frac{dy}{dx}\right]_{x=2}=5(1)^4(12)$$
$$=60$$
Slope of the tangent is 60.

Equation of the tangent at (2, 1) is
$$y-1=60(x-2)$$
$$60x-y-119=0.$$

9. a. $y=3u^2-5u+2$
$$u=x^2-1,\quad x=2$$
$$u=3$$
$$\frac{dy}{du}=6u-5,\quad \frac{du}{dx}=2x$$
$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$
$$=(6u-5)(2x)$$
$$=(18-5)(4)$$
$$=13(4)$$
$$=52$$

d. $y=u(u^2+3)^3,\ u=(x+3)^2,\ x=-2$
$$\frac{dy}{du}=(u^2+3)^3+6u^2(u^2+3)^2 \quad \frac{du}{dx}=2(x+3)$$
$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\left[4^3+6(4)^2\right][2(1)]$$
$$=160\times 2$$
$$=320$$

10. $y=f(x^2+3x-5),\ x=1,\ f'(-1)=2$
$$\frac{dy}{dx}=f'(x^2+3x-5)\times(2x+3)$$
$$=f'(1+3-5)\times 5$$
$$=2\times 5$$
$$\frac{dy}{dx}=10$$

11. $y = g(h(x)),\ h(x) = \dfrac{x^2}{x+2}$

$$\frac{dy}{dx} = g'(h(x)) \times h'(x)$$

When $x = 3,\ h(3) = \dfrac{9}{5}$ and $g'\left(\dfrac{9}{5}\right) = -2.$

$$h'(x) = \frac{(x+2)(2x) - x^2(1)}{(x+2)^2}$$

$$h'(x) = \frac{x^2 + 4x}{(x+2)^2}$$

$$h'(5) = \frac{9+12}{25} = \frac{21}{25}$$

At $x = 3,\ \dfrac{dy}{dx} = -2 \times \dfrac{21}{25}$

$$= -\frac{42}{25}.$$

12. $h(x) = f(g(x))$, therefore $R'(x) = f'(g(x)) \times g'(x)$

$f(u) = u^2 - 1,\ g(2) = 3,\ g'(2) = -1$

Now, $h'(2) = f'(g(2)) \times g'(2)$

$= f'(3) \times g'(2).$

Since $f(u) = u^2 - 1, f'(u) = 2u$, and $f'(3) = 6,$

$\therefore h'(2) = 6(-1)$

$= -6.$

13. $h(x) = p(x)q(x)r(x)$

a. $h'(x) = p'(x)q(x)r(x) + p(x) \times q'(x) \times r(x)$
$+ p(x) \times q(x) \times r'(x)$

14. $y = (x^2 + x - 2)^2 + 3$

$$\frac{dy}{dx} = 2(x^2 + x - 2)(2x+1)$$

At the point $(1, 3)$, $x = 1$ and the slope of the tangent will be $2(1+1-2)(2+1) = 0.$

Equation of the tangent at $(1, 3)$ is $y - 3 = 0$.
Solving this equation with the function, we have

$(x^2 + x - 2)^2 + 3 = 3$

$(x+2)^2(x-1)^2 = 0$

$x = -2$ or $x = 1$

Since -2 and 1 are both double roots, the line with equation $y - 3 = 0$ will be a tangent at both $x = 1$ and $x = -2$. Therefore, $y - 3 = 0$ is also a tangent at $(-2, 3)$.

15. $y = \dfrac{x^2(1-x)^3}{(1+x)^3}$

$$= x^2\left[\left(\frac{1-x}{1+x}\right)\right]^3$$

$$\frac{dy}{dx} = 2x\left(\frac{1-x}{1+x}\right)^3 + 3x^2\left(\frac{1-x}{1+x}\right)^2\left[-\frac{(1+x)-(1-x)(1)}{(1+x)^2}\right]$$

$$= 2x\left(\frac{1-x}{1+x}\right)^3 + 3x^2\left(\frac{1-x}{1+x}\right)^2 - \frac{2}{(1+x)^2}$$

$$= 2x\left(\frac{1-x}{1+x}\right)^2\left[\frac{1-x}{1+x} - \frac{3x}{(1+x)^2}\right]$$

$$= 2x\left(\frac{1-x}{1+x}\right)^2\left[\frac{1-x^2-3x}{(1+x)^2}\right]$$

$$= -\frac{2x(x^2+3x-1)(1-x)^2}{(1+x)^4}$$

16. If $y = u^n$, prove $\dfrac{dy}{dx} = nu^{n-1}\dfrac{du}{dx}$.

For n = 1, y = u and $\dfrac{dy}{dx} = 1u^{1-1}\dfrac{du}{dy} = \dfrac{du}{dx}$, which is true.

Assume the statement is true for $n = k$, i.e., $y = u^k$, then $\dfrac{dy}{dx} = u^{k-1}\dfrac{du}{dx}$.

For $n = k+1$ show, $\dfrac{dy}{dx} = (k+1)u^k\dfrac{du}{dx}$.

Now, $y = u^{k+1} = u \times u^k$.

$$\frac{dy}{dx}=\frac{du}{dx}\times u^k+u\times ku^{k-1}\frac{du}{dx}$$
$$=\frac{du}{dx}\times u^k+k\times u^k\times\frac{du}{dx}$$
$$=\frac{du}{dx}\times u^k\times(k+1)$$
$$=(k+1)u^k\frac{du}{dx}$$

Therefore, if the statement is true for $n = k$, it will be true for $n = k + 1$. Since it is true for $n = 1$, it will be true for $n = 2$, therefore true for all $n \in N$.

17. $f(x)=ax+b,\quad g(x)=cx+d$

$$f(g(x))=f(cx+d)$$
$$=a(cx+d)+b$$
$$=acx+ad+b$$
$$g(f(x))=g(ax+b)$$
$$=c(ax+b)+d$$
$$=acx+bc+d$$

Now, $f(g(x))=g(f(x))$.

$$\therefore acx+ad+b=ccx+bc+d$$
$$ad-d=bc-b$$
$$d(a-1)=b(c-1)$$

If $f(g(x))=g(f(x))$, then $d(a-1)=b(c-1)$.

Review Exercise

4. f. $y=(x-1)^{\frac{1}{2}}(x+1)$

$$y'=(x-1)^{\frac{1}{2}}+(x+1)-\frac{1}{2}(x-1)^{-\frac{1}{2}}$$
$$=\sqrt{x-1}+\frac{x+1}{2\sqrt{x-1}}$$
$$=\frac{2x-2+x+1}{2\sqrt{x-1}}$$
$$=\frac{3x-1}{2\sqrt{x-1}}$$

h. $y=\sqrt{(x+3)(x-3)}=(x^2-9)^{\frac{1}{2}}$

$$y'=\frac{1}{2}(x^2-9)^{-\frac{1}{2}}(2x)$$
$$=\frac{x}{\sqrt{x^2-9}}$$

5. c. $y=\dfrac{(2x-5)^4}{(x+1)^3}$

$$y'=\frac{(x+1)^3\times4(2x-5)^3-3(2x-5)^4(x+1)^2}{(x+1)^6}$$
$$=\frac{(x+1)^2(2x-5)^3[4x+4-6x+15]}{(x+1)^6}$$
$$y'=\frac{(2x-5)^3(19-2x)}{(x+1)^4}$$

f. $y=\dfrac{(x^2-1)^3}{(x^2+1)^3}=\left(\dfrac{x^2-1}{x^2+1}\right)^3$

$$y'=3\left(\frac{x^2-1}{x^2+1}\right)^2\left[\frac{(x^2+1)(2x)-2x(x^2-1)}{(x^2+1)^2}\right]$$
$$=\frac{12x(x^2-1)^2}{(x^2+1)^4}$$

i. $y=(1-x^2)^3(6+2x)^{-3}$

$$=\left(\frac{1-x^2}{6+2x}\right)^3$$
$$y'=3\left(\frac{1-x^2}{6+2x}\right)^2\left[\frac{(6+2x)(-2x)-(1-x^2)(2)}{(6+2x)^2}\right]$$
$$=\frac{3(1-x^2)^2(-12x-4x^2-2+2x^2)}{(6+2x)^4}$$
$$=-\frac{3(1-x^2)^2(2x^2+12x+2)}{(6+2x)^4}$$
$$=-\frac{3(1-x^2)^2(x^2+6x+1)}{8(3-x)^4}$$

6. **a.** $g(x) = f(x^2)$

$g'(x) = f'(x^2) \times 2x$

b. $h(x) = 2xf(x)$

$h'(x) = 2f(x) + 2xf'(x)$

7. **b.** $y = \dfrac{u+4}{u-4}$, $u = \dfrac{\sqrt{x}+x}{10}$,

$x = 4$

$u = \dfrac{3}{5}$

$$\frac{dy}{du} = \frac{(u-4)-(u+4)}{(u-4)^2} \qquad \frac{du}{dx} = \frac{1}{10}\left(\frac{1}{2}x^{-\frac{1}{2}}+1\right)$$

$$= -\frac{8}{(u-4)^2} \qquad \left.\frac{du}{dx}\right]_{x=4} = \frac{1}{10}\left(\frac{5}{4}\right)$$

$$= \frac{1}{8}$$

$$\left.\frac{dy}{du}\right]_{u=\frac{3}{5}} = -\frac{8}{\left(\frac{3}{5}-\frac{20}{5}\right)^2}$$

$$= -\frac{8(25)}{(-17)^2}$$

$$\left.\frac{dy}{dx}\right]_{x=4} = -\frac{8(25)}{17^2} \times \frac{1}{8}$$

$$= \frac{25}{289}$$

c. $y = f\left(\sqrt{x^2+9}\right)$, $f'(5) = -2$, $x = 4$

$$\frac{dy}{dx} = f'\left(\sqrt{x^2+9}\right) \times \frac{1}{2}(x^2+9)^{-\frac{1}{2}}(2x)$$

$$\frac{dy}{dx} = f'(5) \bullet \frac{1}{2} \bullet \frac{1}{5} \bullet 8$$

$$= -2 \bullet \frac{4}{5}$$

$$= -\frac{8}{5}$$

9. $y = -x^3 + 6x^2$

$y' = -3x^2 + 12x$

$-3x^2 + 12x = -12 \qquad -3x^2 + 12x = -15$

$x^2 - 4x - 4 = 0 \qquad x^2 - 4x - 5 = 0$

$x = \dfrac{4 \pm \sqrt{16+32}}{2} \qquad (x-5)(x+1) = 0$

$= \dfrac{4 \pm 4\sqrt{3}}{2} \qquad x = 5, \quad x = -1$

$x = 2 \pm 2\sqrt{3}$

10. **a. i)** $y = (x^3 - x)^2$

$y' = 2(x^3 - x)(3x^2 - 1)$

Horizontal tangent,

$2x(x^2 - 1)(3x^2 - 1) = 0$

$x = 0, \quad x = \pm 1, \quad x = \pm\dfrac{\sqrt{3}}{3}.$

11. **b.** $y = (3x^{-2} - 2x^3)^5$ at $(1,1)$

$y' = 5(3x^{-2} - 2x^3)^4(-6x^{-3} - 6x^2)$

$A + x = 1$

$y' = 5(1)^4(-6-6)$

$= -60$

Equation of the tangent at (1, 1) is

$$y - 1 = -60(x-1)$$

$60x + y - 61 = 0.$

12. $y = 3x^2 - 7x + 5$

$\dfrac{dy}{dx} = 6x - 7$

Slope of $x + 5y - 10 = 0$ is $-\dfrac{1}{5}$.

Since perpendicular, $6x-7=5$

$$x=2$$

$$y=3(4)-14+5$$

$$=3.$$

Equation of the tangent at (2, 3) is

$$y-3=5(x-2)$$

$$5x-y-7=0.$$

13. $y=8x+b$ is tangent to $y=2x^2$

$$\frac{dy}{dx}=4x$$

Slope of the tangent is 8, therefore $4x=8,\ x=2$.

Point of tangency is (2, 8).

Therefore, $8=16+b,\ b=-8$.

Or $\quad 8x+b=2x^2$

$$2x^2-8x-b=0$$

$$x=\frac{8\pm\sqrt{64+8b}}{2(2)}.$$

For tangents, the roots are equal, therefore

$64+8b=0,\quad b=-8$.

Point of tangency is (2, 8), $b=-8$.

15. a. $f(x)=2x^{\frac{5}{3}}-5x^{\frac{2}{3}}$

$$f'(x)=2\times\frac{5}{3}x^{\frac{2}{3}}-5\times\frac{2}{3}x^{-\frac{1}{3}}$$

$$=\frac{10}{3}x^{\frac{2}{3}}-\frac{10}{3x^{\frac{1}{3}}}$$

$f(x)=0\ \therefore x^{\frac{2}{3}}[2x-5]=0$

$x=0$ or $x=\dfrac{5}{2}$

$y=f(x)$ crosses the x-axis at $x=\dfrac{5}{2}$, and

$$f'(x)=\frac{10}{3}\left(\frac{x-1}{x^{\frac{1}{3}}}\right)$$

$$f'\left(\frac{5}{2}\right)=\frac{10}{3}\times\frac{3}{2}\times\frac{1}{\left(\frac{5}{2}\right)^{\frac{1}{3}}}$$

$$=5\times\frac{3\sqrt{2}}{3\sqrt{5}}=5^{\frac{2}{3}}\times2^{\frac{1}{3}}$$

$$=(25\times2)^{\frac{1}{3}}$$

$$=3\sqrt{50}$$

b. To find a, let $f(x)=0$.

$$\frac{10}{3}x^{\frac{2}{3}}-\frac{10}{3x^{\frac{1}{3}}}=0$$

$$30x=30$$

$$x=1$$

Therefore $a=1$.

18. $C(x)=\dfrac{1}{3}x^3+40x+700$

a. $C'(x)=x^2+40$

b. $C'(x)=76$

$$\therefore x^2+40=76$$

$$x^2=36$$

$$x=6$$

Production level is 6 gloves/week.

19. $R(x)=750x-\dfrac{x^2}{6}-\dfrac{2}{3}x^3$

a. Marginal Revenue

$$R'(x)=750-\frac{x}{3}-2x^2$$

b. $R'(10)=750-\dfrac{10}{3}-2(100)$

$$=\$546.67$$

20. $D(p) = \dfrac{20}{\sqrt{p-1}}, \quad p > 1$

$$D'(p) = 20\left(-\frac{1}{2}\right)(p-1)^{-\frac{3}{2}}$$

$$= -\frac{10}{(p-1)^{\frac{3}{2}}}$$

$$D'(5) = -\frac{10}{\sqrt{4}^3} = -\frac{10}{8}$$

$$= -\frac{5}{4}$$

Slope of demand curve at (5, 10) is $-\dfrac{5}{4}$.

Chapter 4 Test

2. f is the graph on the right and below the x-axis (it's a cubric). f' is the other graph (it is quadratic).

3. $f(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{x+h-(x+h)^2-(x-x^2)}{h}$$

$$= \lim_{h\to 0}\frac{x+h-(x^2+2hx+h^2)-x+x^2}{h}$$

$$= \lim_{h\to 0}\frac{h-2hx-h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(1-2x-h)}{h}$$

$$= \lim_{h\to 0}(1-2x-h)$$

$$= 1-2x$$

Therefore, $\dfrac{d}{dx}(x-x^2) = 1-2x.$

4. a. $y = \dfrac{1}{3}x^3 - 3x^{-5} + 4\pi$

$$\frac{dy}{dx} = x^2 + 15x^{-6}$$

b. $y = 6(2x-9)^5$

$$\frac{dy}{dx} = 30(2x-9)^4(2)$$

$$= 60(2x-9)^4$$

c. $y = \dfrac{2}{\sqrt{x}} + \dfrac{x}{\sqrt{3}} + 6\sqrt[3]{x}$

$$= 2x^{-\frac{1}{2}} + \frac{1}{\sqrt{3}}x + 6x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = -x^{-\frac{3}{2}} + \frac{1}{\sqrt{3}} + 2x^{-\frac{2}{3}}$$

d. $y = \left(\dfrac{x^2+6}{3x+4}\right)^5$

$$\frac{dy}{dx} = 5\left(\frac{x^2+6}{3x+4}\right)^4 \frac{2x(3x+4)-(x^2+6)3}{(3x+4)^2}$$

$$= \frac{5(x^2+6)^4(3x^2+8x-18)}{(3x+4)^6}$$

e. $y = x^2\sqrt[3]{6x^2-7}$

$$\frac{dy}{dx} = 2x(6x^2-7)^{\frac{1}{3}} + x^2\frac{1}{3}(6x^2-7)^{-\frac{2}{3}}(12x)$$

$$= 2x(6x^2-7)^{-\frac{2}{3}}\left((6x^2-7)+2x^2\right)$$

$$= 2x(6x^2-7)^{-\frac{2}{3}}(8x^2-7)$$

f. $y = \dfrac{4x^5 - 5x^4 + 6x - 2}{x^4}$

$= 4x - 5 + 6x^{-3} - 2x^{-4}$

$\dfrac{dy}{dx} = 4 - 18x^{-4} + 8x^{-5}$

$= \dfrac{4x^5 - 18x + 8}{x^5}$

5. $y = (x^2 + 3x - 2)(7 - 3x)$

$\dfrac{dy}{dx} = (2x + 3)(7 - 3x) + (x^2 + 3x - 2)(-3)$

At $(1,8)$,

$\dfrac{dy}{dx} = (5)(4) + (2)(-3)$

$= 14.$

The slope of the tangent to $y = (x^2 + 3x - 2)(7 - 3x)$ at $(1,8)$ is 14.

6. $y = 3u^2 + 2u$

$\dfrac{dy}{du} = 6u + 2$

$u = \sqrt{x^2 + 5}$

$\dfrac{du}{dy} = \dfrac{1}{2}(x^2 + 5)^{-\frac{1}{2}} 2x$

$\dfrac{dy}{dx} = (6u + 2)\left(\dfrac{x}{\sqrt{x^2 + 5}}\right)$

At $x = -2, u = 3.$

$\dfrac{dy}{dx} = (20)\left(-\dfrac{2}{3}\right)$

$= -\dfrac{40}{3}$

7. $y = (3x^{-2} - 2x^3)^5$

$\dfrac{dy}{dx} = 5(3x^{-2} - 2x^3)^4(-6x^{-3} - 6x^2)$

At $(1, 1)$,

$\dfrac{dy}{dx} = 5(1)^4(-6 - 6)$

$= -60.$

Equation of tangent line at $(1, 1)$ is

$\dfrac{y - 1}{x - 1} = -60$

$y - 1 = -60x + 60$

$60x + y - 61 = 0.$

8. $P(t) = \left(t^{\frac{1}{4}} + 3\right)^3$

$P'(t) = 3\left(t^{\frac{1}{4}} + 3\right)^2\left(\dfrac{1}{4}t^{-\frac{3}{4}}\right)$

$P'(16) = 3\left(16^{\frac{1}{4}} + 3\right)^2\left(\dfrac{1}{4} \times 16^{-\frac{3}{4}}\right)$

$= 3(2 + 3)^2\left(\dfrac{1}{4} \times \dfrac{1}{8}\right)$

$= \dfrac{75}{32}$

The amount of pollution is increasing at a rate of $\dfrac{75}{32}$ p.p.m./year.

9. $y = x^4$

$\frac{dy}{dx} = 4x^3$

$-\frac{1}{16} = 4x^3$

Normal line has a slope of 16. Therefore, $\frac{dy}{dx} = -\frac{1}{16}$.

$x^3 = -\frac{1}{64}$

$x = -\frac{1}{4}$

$y = -\frac{1}{256}$

Therefore, $y = x^4$ has a normal line with a slope of 16 at $\left(-\frac{1}{4}, \frac{1}{256}\right)$.

10. $y = x^3 - x^2 - x + 1$

$\frac{dy}{dx} = 3x^2 - 2x - 1$

For a horizontal tangent line, $\frac{dy}{dx} = 0$.

$3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0$

$x = -\frac{1}{3}$ or $x = 1$

$y = 1 - 1 - 1 + 1$

$= 0$

$y = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 1$

$= \frac{-1 - 3 + 9 + 27}{27}$

$= \frac{32}{27}$

The required points are $\left(-\frac{1}{3}, \frac{32}{27}\right), (1, 0)$.

11. $y = x^2 + ax + b$

$\frac{dy}{dx} = 2x + a$

$y = x^3$

$\frac{dy}{dx} = 3x^2$

Since the parabola and cubic function are tangent at (1, 1), then $2x + a = 3x^2$.

At (1, 1) $2(1) + a = 3(1)^2$

$a = 1.$

Since (1,1) is on the graph of $y = x^2 + x + b$,

$1 = 1^2 + 1 + b$

$b = -1.$

The required values are 1 and –1 for *a* and *b* respectively.